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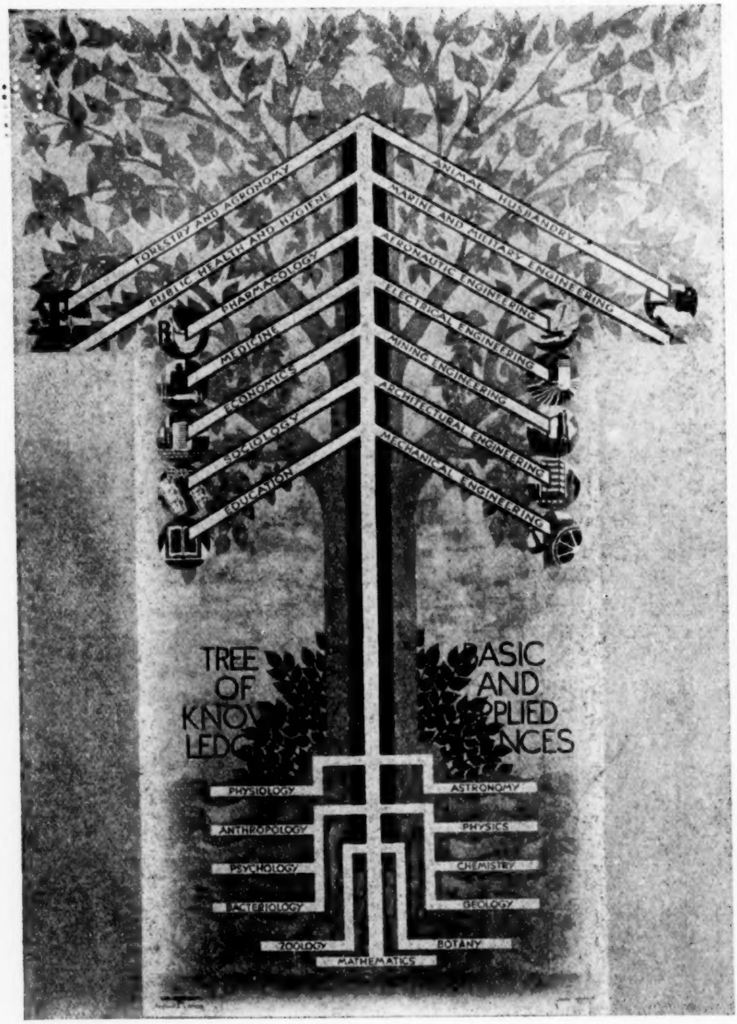
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An Interpretation and Comparison of Three Schools of Thought in the Foundations of Mathematics*

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ONE OF THE notable features of twentieth century mathematics has been the extensive and critical investigation into the foundations of the subject. At the present time there seem to be three schools of thought in the foundations which are of special prominence and interest. These are the *postulational*, the *logical*, and the *formalist* schools.

The postulational school is led in this country by Professors E. V. Huntington and Oswald Veblen. The specific aim of the school is to establish satisfactory sets of postulates for various branches of mathematics.

The logical school centers around Bertrand Russell and Professor A. N. Whitehead, and their three-volume treatise, *Principia*

* This paper expands certain ideas contained in the first section of a previous article by the present writer, entitled "Teaching an Appreciation of Mathematics: The Need of Reorganization in Geometry," THE MATHEMATICS TEACHER, 27:30–40, Jan. 1934. The present treatment, however, is entirely from the mathematical-logical point of view, and the discussion is much more technical and detailed than in the earlier article. This essay received a Bowdoin College prize at Harvard University in May 1934.

Mathematica. The members of this school are interested in the explicit formulation of symbolic logic as a foundation for mathematics.

The formalist school is led by David Hilbert of the University of Göttingen, an eminent mathematician, who near the beginning of the century would have been classed as a postulationist. The formalists are attempting to make mathematical proofs rigorous by formalizing the structure of mathematics.

It is the purpose of the present discussion to do three things: *first,* to explain in some detail the nature of the program of each school of thought; *second,* to interpret the implications of each program for a corresponding point of view toward the nature of mathematics as a whole; *third,* to compare the resulting points of view in an attempt to formulate some generalized conclusions concerning the nature of mathematics.

For guidance in making the interpretations it seems desirable to have some general standard of comparison which is not related to any of the three schools of thought. Such a standard may be furnished by the traditional view of mathematics, which undoubtedly still has considerable popular support. This view may be epitomized as follows: "Mathematics is the science of number and space. It starts from a group of self-evident truths and by infallible deductions arrives at incontestable conclusions. Whereas the facts of other sciences are at best approximate truths continually subject to change in the light of further and more exacting experimentation or observation, the facts of mathematics are absolute, unalterable, and eternal truths."

In the light of this statement, we propose to consider the nature of the subject matter, substantive foundations, method, structure and truth of mathematics, when making our interpretations. It will also be significant to investigate the nature of mathematical symbolism as it appears in each of the programs.

Our procedure will be to devote two sections to each of the schools; the first section will, in general, be expository of the program or specific point of view of the school, while the second section will offer the interpretation for a larger point of view toward mathematics. A final section will give the comparative survey of the different points of view.

I

The reasoning underlying the program of the postulational school is simple, and amounts essentially to this.[1] Any branch of mathematics must have a starting point somewhere. Not all of its propositions can be proved and neither can all of its technical terms be defined. If someone claims to have proved completely all propositions, and to have defined completely all terms, either he must have assumed certain propositions unconsciously and used certain terms glibly without realizing that they were undefined, or else he has been guilty of a "vicious circle" error.

In order to proceed rationally in the development of a mathematical discipline, it seems very desirable to make the unproved propositions, that is, the postulates or axioms, and the undefined terms, as explicit as possible. Then the concepts of the subject can be defined in terms of the undefined concepts, and, by the aid of logical reasoning, we can proceed to deduce further propositions, or theorems, from the unproved propositions. In this way, using the deductive method and the associated process of definition, the structure of the subject can be built up systematically from the foundations. On the other hand, if the foundations are not made explicit, it is conceivable that vicious circles may occur in the subject, or we may find that two propositions have been tacitly taken for granted which are mutually contradictory. In the latter case, we should expect to find contradictions among the theorems deduced from these propositions. To help avoid difficulties of this nature it is necessary to adopt a definite restricted set of postulates and a definite restricted number of undefined terms. At the same time, it is clear that the primary qualification for an acceptable set of postulates must be *logical consistency;* that is, no one of the postulates must contradict any other.

How is the consistency of a tentative set of postulates for a given mathematical discipline established? The answer resides in the nature of the undefined terms.

Let us suppose that in the beginning the tentative postulates are obtained by listing certain fundamental relationships which are assumed or believed to be true in the given subject. Certain

[1] The present section contains a general discussion of a subject often treated in the literature of mathematical philosophy and postulational research. For selected references, see the Bibliography.

undefined terms have also been selected, and presumably are prominent in the statement of the postulates. At this stage, the undefined terms and the postulates have some concrete or psychological significance to the mind. For example, the postulates may make concrete statements about such undefined terms as points and lines, or numbers.

But if the undefined terms are really recognized as undefined, it must be possible to abstract all previous connotations from them, and to treat them as mere symbols, devoid of any special significance to the mind other than what may be implied about them in broad preliminary descriptions, or in the statement of the postulates. It must be equally possible to reinterpret these symbols in new ways, which have concrete significance different from the preconceived significance of the original undefined terms. With a new interpretation of the symbols each postulate of the set will again make a concrete statement, and, if the interpretation is suitable, this can be judged by the mind to be true or false. Each postulate is then said to be satisfied, or not satisfied, for the particular interpretation in question. If some new concrete interpretation can be found—which itself appeals to our judgment as being self-consistent—such that *all* of the postulates are judged to be satisfied, then it is claimed that the postulates actually are logically consistent. For otherwise, contradictory assertions would have to be true about this concrete interpretation which is held to be self-consistent.

As a rough illustration we may refer to a method of establishing the consistency of a special type of elliptic (non-Euclidean) plane geometry. Let us start with an undefined class of elements called "points"; an undefined sub-class of points called "straight lines," or simply "lines"; and an undefined number associated with two points of a line, called "distance" or "length." We assume knowledge of the logical notion of "class of elements," and an "element belonging to a class"; and we also assume knowledge of certain ideas of arithmetic and general language. A point, P, is said to *lie on* a line, *l*, or the line to *pass through* the point, provided P belongs to the particular sub-class of points which constitutes *l*. Two lines are said to *intersect* if there is a point which belongs to both of them. With these preliminary agreements, let us propose the following postulates, among others, for our two-dimensional geometry:

1. Two distinct "lines" intersect in two and only two distinct "points."
2. Through the two intersection "points" of two "lines" there pass an infinite number of "lines."
3. The "distance" between the two intersection "points" of two "lines" is the same along all of the "lines" which pass through the two "points."
4. Every "line" has a finite "length" which is equal to the "length" of every other "line."
5. Through two "points" which are not intersection "points" of two "lines" there passes one and only one "line."

These postulates are not altogether easy to comprehend, and a person thoroughly imbued with the traditional view of mathematics would not hesitate to deny their validity, even if reminded that "point," "line", and "length" are undefined terms. But "point," "line", and "length" *are* undefined, and we therefore have a right to treat them as abstract notions without regard to any of our preconceived ideas about them. To emphasize this we might replace "point," "line," and "length" wherever they occur in the postulates by nonsense syllables, such as "fi," "fo," "fum." We should then be in a position to reinterpret the words "fi," "fo," "fum" in whatever way we might choose. However, it will be simpler to remember the abstractness of the original terms used in the postulates, and to reinterpret these words directly. Our object is to find some concrete interpretation which will satisfy all five postulates, and it is very easy to do this. We shall interpret our class of "points" as the class of *points on the surface of a fixed sphere* in Euclidean geometry of three dimensions; we shall interpret a "line" as a *great circle* of this sphere; and we shall interpret "length" or "distance" as the usual concept of *arc length* or *distance* measured along a great circle of the sphere. Our postulates now read as follows:

1′. Two distinct great circles of the sphere intersect in two and only two distinct points of the sphere.
2′. Through the two intersection points of two great circles of the sphere there pass an infinite number of great circles.
3′. The distance between the two intersection points of two great circles of the sphere is the same along all of the great circles which pass through the two points.

4′. Every great circle has a finite length which is equal to the length of every other great circle.

5′. Through two points of the sphere which are not intersection points of two great circles there passes one and only one great circle.

These statements are all quickly judged to be correct because of our previous acquaintance with the Euclidean geometry of the sphere. Furthermore, since we believe that Euclidean geometry is self-consistent, we can now state that our original five postulates are consistent. Or, to make a more conservative claim, our original postulates are *just as consistent* as Euclidean spherical geometry is. We now believe that there is no logical reason why the usual points and straight lines of a plane should not behave as indicated in our original postulates, and we also feel very sure that we can build up a self-consistent system of plane geometry using these postulates.

When a tentative list of postulates has been proposed for some branch of mathematics and has already been shown to be a consistent set, it is perfectly conceivable that certain postulates of the set are logically deducible from others of the set. Such postulates are superfluous, or *redundant.* There is no inherent logical fallacy in using a redundant consistent set of postulates, but for at least two reasons it is often desirable that the postulates be free of redundancies, or *independent. First,* an independent set of postulates renders the structure of the subject more aesthetically pleasing; no statement is included as a postulate which might be deduced as a theorem. *Second,* if the redundant postulates are removed, it is possible to go back to any concrete interpretation used in establishing consistency and have fewer postulates to judge true or false than previously. Thus the soundness of the structure of the subject is made to depend more on abstract logical relations, and less on concrete interpretative judgments.

The method used to prove the independence of a set of postulates is as follows. Suppose there are n postulates numbered (1), (2), (3), $\cdots$, (n). If we can find a concrete interpretation of the undefined terms such that all of the postulates (2), (3), $\cdots$, (n) are satisfied, but postulate (1) is *not* satisfied, then (1) is held to be independent of all the others. For if (1) were a logical consequence of some or all of the others it would have to be true for any interpretation for which the others were all true (assuming again that

the interpretation must be self-consistent). Thus, in order to establish the independence of the entire set it is necessary to exhibit n suitable concrete interpretations or examples, using each example to prove the independence of one corresponding postulate.

Another characteristic which a consistent set of postulates may or may not possess is that of *categoricalness* (sometimes referred to as *sufficiency* or *completeness*). Briefly, a set is categorical if it forms the foundation for essentially only one branch of mathematics, while a set is non-categorical if it can serve as a foundation for two or more essentially different branches of mathematics. More explicitly, let us suppose that several concrete systems have been found which satisfy a given set of abstract postulates. Then the set is categorical if a complete "one-to-one correspondence" can be established between the elements and relations of any two concrete systems which satisfy all of the postulates. In this case, any property which is true in the one system, must have a corresponding property in the other. In other words, the postulates suffice to determine the nature of any concrete systems which satisfy them. The postulates together with the theorems which are deduced from them may then be said to constitute a *distinctive* mathematical science. If the undefined terms are considered as abstract, the science retains its abstract character throughout. If the undefined terms are interpreted in various ways so as to obtain concrete systems satisfying all of the postulates, the abstract science is replaced by a number of concrete sciences all of which are really equivalent to each other, and each of which is representative of the underlying abstract science.

It would hardly be possible here to give a satisfactory illustration of a categorical set of postulates, but it is not difficult to cite an example of a non-categorical set, and this will serve our purpose just as well. Let us start with an undefined class, K, of elements which may be designated as $A, B, C, \cdots$. Let us suppose that an undefined operation or relationship between any two elements of the class defines a third element which may or may not belong to the class. The element obtained by operating on A with B we designate as $A\#B$. Let us now agree to the following postulates, known as postulates for a *group*, using "$=$" to indicate that two elements are the same (and noting that different letters do not necessarily refer to different elements):

1. If A and B are elements of the class K, $A\#B$ belongs to K.

2. If A, B, C belong to K, then $(A\#B)\#C = A\#(B\#C)$.
3. There exists a unique element X of K such that $X\#X = X$.
4. For any element A of K there exists a unique element A' of K such that $A\#A' = X$.

The consistency of these postulates may be verified by a multitude of concrete interpretations of the undefined class and operation. One of the simplest examples is to let K represent the class of all rational numbers (positive, negative, and zero) and the operation represent ordinary addition. We judge that the postulates are all satisfied, and find that the element X is the number 0. Again, if we let K represent the class of all rational numbers with the exception of 0, and if we take the operation as ordinary multiplication, the postulates are also satisfied, but now "X" is the number 1. In each of these cases, the elements of a concrete system that satisfies the postulates are infinite in number.

But there are also many concrete systems having only a finite number of elements, which satisfy the postulates. As an example we need only mention a class of two *permutations* on two letters x and y. One permutation, A, replaces x by y and y by x; the other permutation, B, replaces x by itself and y by itself. $A\#B$ here will be interpreted as the result of performing permutation A and following it by permutation B.

Now if we interpret our postulates in accordance with these agreements, the first postulate requires that the result of performing any two permutations of the class successively is a permutation belonging to the class. By trying all the possibilities it is easy to see that this postulate is satisfied. Thus, since A replaces x by y and B replaces y by itself, $A\#B$ replaces x by y and in the same way it replaces y by x; in other words, $A\#B$ is the element A which is known to belong to the class. Similarly we see that $A\#A = B$, $B\#A = A$, and $B\#B = B$. By the same kind of observations we see that the second postulate is satisfied.

The third postulate requires that a unique "identical" element, X, exists such that $X\#X = X$. The element B meets this requirement inasmuch as $B\#B = B$; furthermore, it is the only element of K which meets the requirement, as A does not meet it.

The fourth postulate requires the existence of unique "inverse" elements, A' and B', for both A and B. Now B is the identical element and we recall that $A\#A = B$, $B\#B = B$; furthermore, if we operate on A by B, or on B by A, we do not obtain the identical

element. Thus, there is a unique inverse for each element of the class—namely, the element itself—and the postulate is satisfied.

We can now judge that this concrete class of two elements (permutations), with the accompanying interpretation of the undefined operation, satifies all four of our postulates. But we have already indicated that systems which satisfy the postulates may contain an *infinite* number of elements. It is obvious that we cannot set up a one-to-one correspondence between the elements (and operations) of a system containing two elements, and the elements (and operations) of a system containing an infinite number of elements. Hence the four postulates are non-categorical; they are not sufficient to determine a distinctive mathematical science.

It is not to be inferred that a set of postulates is not useful if non-categorical. On the contrary, there are often advantages in having a non-categorical set. For in this way it is possible to develop parts of a number of separate branches of mathematics at the same time. Thus, in any system which satisfies the postulates for a group, the theorems which can be deduced from these postulates will be true, regardless of whether or not the systems can be put into one-to-one correspondence with each other.

In summarizing the important features of the program of the postulational school, we can now make the following statements. The first concern of the school is to establish consistent sets of postulates for various mathematical sciences. It is usually desirable that a set of postulates be independent, and sometimes a set is desired to be categorical, sometimes non-categorical. It is notable that in establishing consistency, independence, and categoricalness, the proofs depend *first*, on the abstract nature of the postulates when the undefined terms are treated as abstract symbols; *second*, on the possibility of interpreting the undefined terms, and hence the postulates, in many concrete ways having psychological or intuitive significance; *third*, on a process of judging that these postulates are satisfied or not satisfied for a given concrete interpretation or system; and *fourth*, on an assumption that each of the concrete systems used is self-consistent.

II

What is the significance of the program of the postulational school for a larger point of view toward mathematics?

It is clear, in the first place, that the nature of specific mathe-

matical *subject matter* is a subordinate problem from the postulational standpoint. A set of postulates, once shown to be consistent, can be used either as the starting point of a purely abstract mathematical science, developed logically from the postulates in their abstract form, or of a number of concrete mathematical sciences dealing with more than one field of subject matter. No specific type of concrete subject matter has a right to be given priority as *the* subject matter of a given branch of mathematics. The traditional notions that "number and space" constitute the sole subject matter of mathematics are thus discarded.

Next it should be noted that from the postulational standpoint, mathematics is not one complete deductive science, but rather is composed of numerous mathematical sciences each of which has its own starting point consisting of undefined terms and postulates, and each of which deduces logically its own set of theorems from its own set of postulates. A mathematical science thus exhibits both the deductive method of reasoning and a deductive structure of thought. The deductive method is the active aspect of building up the deductive structure.

But what is the fundamental nature of this deductive process of reasoning? By what right do we combine two or more postulates to arrive at a theorem? Into such questions the members of the postulational school have not entered at much length. They do commonly seek to establish sets of postulates for the "algebra of logic," and these postulates, when properly interpreted, become concrete statements about relationships of propositions—that is, about logical reasoning.[2] But this algebra in its abstract form, at least, is viewed only as representative of a *branch of mathematics* making use, itself, of principles of logical deduction, which remain outside of itself. These principles seem to remain largely unanalyzed and taken for granted.

Now let us look in more detail at the nature of the foundations or starting point of a mathematical science. Two questions are of special interest: First, to what extent can the consistency of a set

[2] See, for example, Bernstein, B.A. "Whitehead and Russell's Theory of Deduction as a Mathematical Science." *Bulletin of the American Mathematical Society*, 37: 480–488, June 1931.

Also Huntington, E. V. "New Sets of Independent Postulates for the Algebra of Logic, with Special Reference to Whitehead and Russell's *Principia Mathematica*." *Transactions of the American Mathematical Society*, 35: 274–304, Jan. 1933.

of postulates be established? Second, are the postulates ever to be considered "self-evident truths?"

It seems quite evident that consistency cannot be absolutely established by the methods of the postulational school. For, in each consistency proof we have seen that an assumption is involved to the effect that a certain concrete system, interpreting the undefined terms, is self-consistent. Sometimes it may be possible to justify the assumption temporarily by showing that the particular concrete interpretation is self-consistent if some other system is self-consistent; but obviously such a process of shifting responsibility cannot be continued indefinitely without circularity, or an unverified assumption somewhere that a certain concrete system is self-consistent. Usually it seems simpler not to undertake such a process of shifting responsibility beyond the first step of saying "this set of postulates is consistent, on the assumption that this concrete system is self-consistent." Often (as in the case of the postulates for a group) there are numerous concrete systems which are judged to satisfy a given set of postulates. This situation tends to strengthen the evidence in favor of consistency to such a degree that consistency may be said to be almost certain, but nevertheless the proof is not absolute.

The second question concerning the foundations has to do with the closely associated problem of the sense in which the postulates are to be considered true. In particular, are they ever to be considered "self-evident truths?"

Obviously, if we are dealing with an abstract science there is no significance in inquiring into the truth or falsity of the abstract postulates on which the science depends. For, the postulates lack concrete meaning and hence cannot be judged true or false as they stand. To be sure, they are required to be consistent, and hence must be judged to be true for at least one concrete interpretation of the abstract undefined terms; but this judgment serves a purely instrumental purpose here. By no stretch of the imagination could the postulates underlying an abstract, mathematical science be called "self-evident truths."

What happens when the postulates are considered definitely in relation to a concrete system interpreting the undefined terms, as is the case when proving consistency, independence, or categoricalness? The postulates then seem to be viewed as little more than *criteria* which a given system may or may not satisfy. To call these

criteria "self-evident truths" is obviously absurd. But once a concrete system has been found which is judged to satisfy all of the postulates, may we not say that the affirmations that the postulates hold for this system constitute "self-evident truths?" The best way to answer this question is to note that a concrete interpretation of the undefined terms cannot serve to define these terms rigorously; for if this were the case we should have an example of a concrete science in which all terms were defined. The concrete interpretations, in the last analysis, serve only to give the undefined terms intuitive or psychological meaning. In judging that the concrete system satisfies all of the postulates, we are simply relying, then, on our intuitive notions about the concrete system. The actual process of judging may involve direct use of intuition or observation; or it may involve a presupposed acquaintance with some other branch of mathematics (as in the case of judging that Euclidean spherical geometry satisfies the postulates we proposed for elliptic plane geometry); or it may involve substantial processes of reasoning. Sometimes the verdict may *seem* to be self-evident; sometimes it may seem very far from self-evident. But in any case, the strongest statement justified in claiming that the postulates are satisfied for a concrete system would appear to be this: *the postulates seem to be satisfied on the ultimate basis of our best intuitive judgment.*

Incidentally, this same consideration strengthens our previous assertion that a consistency proof is not absolute; and the same applies to independence proofs. Both kinds of proof depend on elements of intuitive judgment, as well as on an assumption of self consistency for a certain concrete system.

It should be noted that sometimes a set of postulates is shown by the usual procedure to satisfy one concrete interpretation, and thus to be consistent, and then is used as a basis for a concrete science involving a different concrete interpretation, even though the postulates seem intuitively to be false for this other interpretation. Such was precisely the case in our illustration of the postulates for elliptic plane geometry. When the abstract terms "point" and "line" were interpreted as point and great circle of a sphere the postulates were judged to be satisfied, and thus consistent. It was then agreed that there is no logical reason why they cannot be used as a basis for a concrete science concerned with the usual concepts of point and straight line in the plane. The postulates do not

seem to be intuitively true for this interpretation, but inasmuch as they are shown to be just as consistent as Euclidean geometry they are *assumed* as a starting point for a certain new kind of plane geometry, which is logically just as valid as Euclidean geometry.

This suggests that the most inclusive way of viewing the postulates—whether concrete or abstract—is to think of them as mere assumptions, or agreements as to fundamental properties, which are made for the purpose of getting started in a particular branch of mathematics. Once the postulates are shown (by a non-absolute proof) to be consistent, it does not matter what degree of truth is attributed to them in any concrete form. They are simply hypotheses for the whole subject. The subject is built up by saying "*if* the postulates are true, *then* these theorems are true."

This last statement suggests the attitude which must be taken toward the truth of the *theorems* of a concrete or abstract mathematical science. They certainly are not absolutely and eternally true inasmuch as the postulates are not. The most we have a right to say is that the theorems are true relative to the original postulates. Actually, the qualification must be made still stronger when we remember that the theorems are obtained by use of deductive reasoning and that the methods of deductive reasoning have been simply assumed to be sound. Thus the verdict must be that the theorems are true at most in relation to the postulates, and to the methods of reasoning used in deducing the theorems from the postulates.

It now seems possible to summarize as follows the characteristic features of mathematics from the postulational point of view: Mathematics is a collection of mathematical sciences whose subject matter may be considered either as abstract, or concrete in innumerable directions. Any mathematical science, in completed form, is a deductive structure of thought exhibiting a logical chain of reasoning from postulates to theorems, and a corresponding building up process from undefined terms to defined terms. The postulates are not to be considered as self-evident truths, but rather as assumptions concerning fundamental properties which are made in the beginning for the purpose of getting started in the particular branch of mathematics under consideration. It is essential that the postulates be consistent, but absolute proofs of consistency do not seem to be possible. The theorems are not absolutely true, but rather are true at most in relation to the

postulates and the methods of deductive reasoning used in deriving them.

III

In considering the logical school, the discussion will be based largely on the *Principia Mathematica* of Whitehead and Russell.

The entire program of the *Principia* may be considered an attempt to justify the authors' fundamental thesis that *mathematics is an extension of formal logic*. This thesis is well stated in the following quotation from Russell. Referring to mathematics and logic he says:[3]

> . . . It has now become wholly impossible to draw a line between the two; in fact, the two are one. They differ as boy and man: logic is the youth of mathematics and mathematics is the manhood of logic. . . . The proof of their identity is, of course, a matter of detail: starting with premisses which would be universally admitted to belong to logic, and arriving by deduction at results which obviously belong to mathematics, we find that there is no point at which a sharp line can be drawn with logic to the left and mathematics to the right.

In other words, mathematics is claimed to be a development out of logic—in terms of *logical concepts*, from *logical principles* and by use of *logical principles*.

In more detail, the program of the *Principia* is as follows. The starting point is a set of undefined or "primitive ideas" and a group of unproved "primitive propositions" of logic, whose choice is held to be more or less an arbitrary matter. A preliminary symbolism is adopted for most of the primitive ideas, and most of the primitive propositions are stated thereby in complete symbolic form. It is significant that the primitive ideas and the corresponding symbols are *not abstract* in the sense that the undefined terms or symbols of a branch of mathematics can be considered to be abstract in the postulational school; on the contrary, in the *Principia* the symbols are used from the beginning to represent concrete logical ideas in concise and convenient form.

No attempt is made to prove the consistency or independence of the primitive propositions, for in the authors' opinion the usual methods of the postulational school do not apply to the fundamentals of logic. The primitive propositions are simply adopted as *true* without further question.

The primitive propositions are used not merely as a substantive

[3] Russell, Bertrand, *Introduction to Mathematical Philosophy*. London, G. Allen and Unwin, and New York, The Macmillan Co. 2nd. ed. 1924, pp. 194–195.

foundation for the deduction of a body of theorems, but also as furnishing the rules of procedure by which theorems may be deduced.

As in the postulational school, concurrent with the deducing of theorems from the primitive propositions is the process of combining the primitive concepts into new concepts through definitions. From the formal point of view, according to Whitehead and Russell, a definition is merely the substitution of a new symbol for a combination of symbols whose meaning is already known on account of representing primitive ideas or on account of having been previously defined. The authors hold that the defining process is theoretically superfluous, but from the practical standpoint it is surely indispensable and coördinate in importance with the deductive process.

These generalized statements may be made more significant by considering a few of the leading primitive ideas and primitive propositions of the *Principia*, together with some important theorems in the theory of deduction (the first section of Volume I).

Important among the primitive ideas are the following: *elementary propositions*, *elementary propositional functions*, *assertion*, *negation*, and *disjunction*. An elementary proposition (designated by p, q, r, and so on) is a statement of the form "this book is green"; an elementary propositional function is a statement with a variable or undetermined element such that when a definite meaning, or value, is assigned to the variable the resulting statement is an elementary proposition. For example, "x is a man" is a propositional function, because if for x we substitute "this Mr. Brown" or "this dog" the result is an elementary proposition. (The concept substituted for the variable must belong, of course, to a suitable class of concepts, in order to obtain a meaningful proposition.) A proposition, p, may be asserted to be true (written "$\vdash p$") or it may be merely considered. The negative of a certain proposition p is the proposition "not-p," or "p is false," written with the aid of the "curl" as "$\sim p$." The disjunction of two propositions p and q is the proposition "p or q," that is, "either p is true or q is true," written with the "wedge" symbol as "$p \vee q$."

The authors use the ideas of elementary proposition, negation, and disjunction to make an all-important definition, namely, the definition of *implication*. The statement "p implies q," written by use of the "horseshoe" symbol as "$p \supset q$," is defined to mean the

same thing as "$\sim p \vee q$," that is, "either p is false or q is true." Informally, the authors give the opinion that this definition agrees with the common use of the phrase "p implies q" in the sense that "*if* p is true, *then* q is true."[4]

The notion of implication is prominent in the statement of the primitive propositions. A few of the more significant of these may be indicated as follows. The actual notation used has been modified by omitting the assertion signs which precede all the primitive propositions (to indicate that they are asserted to be true), and by using parentheses as in ordinary algebra for purposes of grouping, instead of the system of dots used in the *Principia.*

1.1 Anything implied by a true elmentary proposition is true.

1.11 (Paraphrased) If one propositional function, f_1, can be asserted to be true and we can assert that $f_1 \supset f_2$, then we can assert that f_2 is true.

1.2 $(p \vee p) \supset p$. That is, the proposition "p or p" implies the proposition "p."

1.3 $q \supset (p \vee q)$ (Proposition "q" implies the proposition "p or q")

1.4 $(p \vee q) \supset (q \vee p)$ (The proposition "p or q" implies the proposition "q or p").

More strictly speaking, the primitive propositions should be referred to as assertions of primitive propositional functions, for the letters p, q stand for variable or undetermined elementary propositions, and it is asserted that the statements hold for every specific proposition which may be substituted for p, q.

Some of the significant theorems in the theory of deduction which are deduced from the primitive propositions are the following (all of which, again, are understood to be preceded by the assertion sign):

2.01 $(p \supset \sim p) \supset \sim p$. This is the principle of *reductio ad absurdum.* Using the authors' informal interpretation of "implies" as corresponding to "if . . . then," this theorem says: "if p implies not-p, then p is false."

[4] This interpretation has been subject to adverse criticism, and it has been pointed out that the "official" *Principia* development is in no way connected with the "unofficial" interpretations which appear at intervals throughout the work. In particular, to avoid confusion it has been suggested that "$p \supset q$" be read "p horseshoe q" and that it be interpreted always in terms of the definition given for the symbol. See Sheffer, H.M. "Principia Mathematica," *Isis,* 8: 226–231, Feb. 1926.

2.05 $[q \supset r] \supset [(p \supset q) \supset (p \supset r)]$ This is one half of the principle of the syllogism. It is interpreted to say: "If q implies r, then if p implies q, p implies r."

2.11 $p \vee \sim p$. This is the *law of the excluded middle:* "p is true or p is false."

2.12 $p \supset \sim(\sim p)$. This is part of the principle of double negation: "p implies that not-p is false."

2.16 $(p \supset q) \supset (\sim q \supset \sim p)$. "If p implies q, then not-q implies not-p." In other words, the opposite converse of a true proposition of the form "p implies q" is true.

All of these theorems are seen to correspond to methods of deductive reasoning which are usually taken for granted.

Two examples may be given to illustrate the actual methods of proof used in the theorems. First, let us try to prove the principle of *reductio ad absurdum* (2.01 above), namely that $(p \supset \sim p) \supset \sim p$. The steps in the proof are these:

By substituting the proposition $\sim p$ for p in primitive proposition 1.2 the following may be asserted

$$(\sim p \vee \sim p) \supset \sim p$$

But by definition of implication we know that

$$\sim p \vee \sim p \text{ means } p \supset \sim p,$$

so the first assertion is the same as asserting

$$(p \supset \sim p) \supset \sim p, \qquad \text{q.e.d.}$$

Implicitly involved in the proof is the notion that the primitive proposition 1.2 holds good when $\sim p$ is substituted for p in its statement, and similarly that the definition of implication (namely, that $p \supset q$ means the same as $\sim p \vee q$) holds good on substituting $\sim p$ for q. The principle is mentioned but is not formulated officially in the *Principia.*

As a second example, let us consider in detail the method of establishing a certain step in the proof of the opposite-converse proposition, 2.16, above. We wish to prove the assertion that $[p \supset q] \supset [p \supset \sim(\sim q)]$. The procedure is as follows:

By virtue of 2.12, the principle of double negation, we may assert that

$$q \supset \sim(\sim q) \qquad (1)$$

By use of the principle of the syllogism, 2.05, we may assert that

$$[q \supset \sim(\sim q)] \supset [(p \supset q) \supset (p \supset \sim(\sim q)] \qquad (2)$$

Now we note that step (1) asserts that a certain elementary propositional function is true; the statement is true for *any* "value" of the undetermined, variable proposition q. Similarly step (2) asserts that another elementary propositional function, involving two variables, p and q, is always true; more than that, it asserts definitely that the propositional function of step (1) which recurs here in the bracket to the left of the main horshoe, *implies* the propositional function in the right-hand bracket. We are now able to apply the rule given in primitive proposition 1.11, which, in paraphrased form, says: "if one propositional function f_1 can be asserted to be true and we can assert that $f_1 \supset f_2$, then we can assert that f_2 is true." As a result of this rule we can now assert at once the right-hand bracket of step (2), namely:

$$(p \supset q) \supset [p \supset \sim(\sim q)] \qquad \text{q.e.d.}$$

The fundamental importance of primitive proposition 1.11, whose use is illustrated in this proof, probably cannot be overemphasized. In this connection, two statements of the *Principia* are significant: "All the assertions in the present work, with a very few exceptions, assert propositional functions, not definite propositions";[5] and "the above proposition 1.11 is used in every inference from one asserted propositional function to another."[6]

Thus is the tremendous logical-mathematical system of the *Principia Mathematica* developed. The magnitude of the task can be somewhat appreciated by noting that Volume I is wholly devoted to mathematical logic and a "prolegomena to cardinal arithmetic." The number 1 is reached and defined in the middle of Volume I, but not until the beginning of Volume II do we find the long-awaited definition of cardinal number.

IV

Let us now consider the implications of the program of the logical school, as represented by *Principia Mathematica,* for a view of mathematics.

There is surely no quarrel here with the postulational view that the method and structure of mathematics are deductive. But in the postulational school we have seen that the nature of deductive

[5] Whitehead, A. N. and Russell, B. *Principia Mathematica*, Vol. I, 2nd ed. Cambridge Univ. Press, 1925, p. 93.

[6] *Ibid.*, p. 96.

reasoning remains largely unanalyzed, while in the logical school the deductive methods and concepts are themselves developed in great detail from a foundation of undefined terms and unproved propositions of logic. Furthermore, instead of viewing the subject matter of mathematics as wholly abstract, on the one hand, or concrete in innumerable directions on the other, the logical school looks upon mathematical subject matter as consisting of any concepts which may be ultimately traced back and defined in terms of the undefined concepts of logic. Mathematics is not now a collection of deductive sciences, each with its own foundation, but a single, unified, deductive science with a single foundation in logic.

What is the nature of mathematical truth from the standpoint of the logical school? The present writer is not clear as to the specific attitude of the authors of the *Principia* on this question. However, it is hard to see how an impartial judgment could change the verdict, reached from the postulational standpoint, that mathematical truth is *relative*, not absolute. The fact is that the entire *Principia* does rest on *unproved propositions*, and the fact that these propositions specifically concern logical ideas does not make them any the less assumptions, whether or not they are held to be true. This statement would be justified even if there were agreement among logicians that the *Principia* is absolutely perfect. Such an opinion is most certainly lacking.[7]

Finally, it should be noted that the *Principia* furnishes an excellent example of the rôle of symbolism in mathematics. It is natural that symbolism should be more important than in the postulational school, for the postulational school is interested primarily in the formulation of satisfactory sets of postulates, while in the logical school the emphasis is on active development of logic and the fundamentals of mathematics from a set of postulates. In the logical school the symbols for undefined ideas, such as $\vdash$, $\sim$, $\vee$, all have concrete meaning from the beginning. As time goes on, new symbols are defined in terms of the original ones, and the later symbols may tend to become psychologically further removed from concrete thought than the earlier ones. But actually any symbol, such as the symbol for 1, or cardinal number, can always be

[7] Confer, for example, footnote 4 and the confusion between the official and unofficial *Principia*. Another pertinent point is that the authors introduce later in the work a fundamental axiom, the "axiom of reducibility," whose nature, they themselves admit, is a little dubious.

traced back to the original concrete undefined ideas.[8] The value of relying so much on symbols is that it would soon become almost impossible to think about the complex groups of ideas they represent without using them. Furthermore, even if it were possible to do so there would always be danger of misconceptions and preconceived notions inherent in the use of words.[9] The logical school, then, exhibits symbolism as a necessary and economical instrument of thought.

V

Finally we come to Hilbert and the formalist school. Like the logical school, the formalists are attempting to carry the ultimate foundation of mathematical knowledge further back than the postulational school. At the same time, they are trying to establish the consistency of all mathematics, and are thus attacking a problem which is not explicitly investigated in the *Principia*. This much seems certain. But if we seek an exact understanding of Hilbert's methods, and a correct interpretation of their significance for mathematics, we are on more uncertain ground. This is due partly to the incomplete and rather fragmentary form in which the formalist program has thus far been presented, and partly to the fact that there seem to be few authoritative interpreters of Hilbert in this country.

Some interpreters assert that Hilbert views mathematics as a "game played with meaningless marks on paper." For example, E. T. Bell writes:[10]

> Against all the senseless rhetoric that has been wafted like incense before the high altar of "Mathematical Truth," let us put the considered verdict of the man whom most professional mathematicians would agree is the foremost living member of their guild. Mathematics, according to David Hilbert (1862–), is a game played according to certain simple rules with meaningless marks on paper. . . . The *meaning* of mathematics has nothing to do with the game and mathematicians pass outside their proper domain when they attempt to give the marks meanings.

On the face of it, such an interpretation can hardly fail to appear unreasonable, especially in view of Hilbert's productivity as a

[8] See, for example, Lewis, C. I. *Survey of Symbolic Logic*, Berkeley, University of California Press, 1918, pp. 346–349.

[9] For the statement of the case by Whitehead and Russell, see the *Principia*, 2nd. ed. p. 2.

[10] Bell, E. T. *The Queen of the Sciences*. Baltimore, Williams and Wilkins Co., 1931, p. 21.

mathematician. Before proceeding with an account of the formalist program, let us turn to Hilbert's own writings in order to determine, if possible, the extent to which the interpretation seems justified.

For this purpose, we shall refer to a monograph entitled *Die Grundlagen der Mathematik.*[11] This is the latest of Hilbert's general expositions of the subject which the writer has been able to locate.[12] We shall consider each of Bell's implications in turn, together with pertinent statements by Hilbert.

First, Bell implies that Hilbert is not interested in mathematical truth. But in the opening paragraph of *Die Grundlagen* Hilbert asserts that his aim is to solve completely the problems in the foundations of mathematics so as to make mathematical propositions irrefutable ("unwiderleglich"). To accomplish this aim he has formulated a Theory of Proof ("Beweistheorie"), in which he tries to express every mathematical theorem as a strictly deductive formula ("streng ableitbaren Formel").

Second, whereas according to Bell, it is Hilbert's "considered verdict" that "mathematics is a game," in the monograph cited, Hilbert makes no mention of mathematics as a game until answering a criticism by Brouwer to the effect that the formalists make mathematics degenerate into a game ("ein Spiel ausarte"). Hilbert replies, in effect, that the formalist program may be called a game if so desired, but indicates that it is a game which has a more serious purpose, both for mathematics and philosophy, than the mere playing of it.

Third, Bell asserts that the game is played with "meaningless marks." But Hilbert distinguishes between what he calls real propositions and ideal propositions; and the real propositions are definitely stated to have meaning, to be directly verifiable. The ideal propositions and axioms of the *Beweistheorie,* all stated as formulas, are asserted to be the image of the thoughts belonging to the usual development of mathematics, but ***in themselves*** ("an sich"), Hilbert says, the formulas, and the marks which are used in the formulas, are to have no meaning. These considerations do not seem to justify the flat statement that the marks are meaningless.

[11] Hilbert, D. *Die Grundlagen der Mathematik.* Leipzig, B. G. Teubner, 1928.

[12] Since writing this article a new and comprehensive discourse has appeared, namely: Hilbert, D. and Bernays, P. *Grundlagen der Mathematik.* Erster Band. Berlin, J. Springer, 1934.

Fourth, Bell's statement that "certain simple rules" are used in the game, seems to imply that the rules used by Hilbert are arbitrarily chosen, but Hilbert definitely indicates that his rules are *not* arbitrary, in fact that they are meant to correspond to our usual methods of thinking. The nature of the rules will be indicated in more detail later.

For these specific reasons, an interpretation of the formalist school like the one offered by Bell seems indeed to be a misinterpretation.[13] Let us now investigate the nature of the formalist program in more detail, with the hope of reaching a more reasonable interpretation. We shall continue to rely largely on *Die Grundlagen*.

It is Hilbert's contention that the ultimate foundation for mathematics lies not in logic, but rather in certain *pre-logical* objects—marks or symbols—which are preliminary conditions for logical thinking, and about which we seem to have definite intuitive knowledge.

Certain mathematical statements, made by the use of symbols, are immediately capable of verification by intuitive methods, because of the inherent ("inhältliche") nature of the concepts represented by the symbols, for example

$$3+1=1+3, \qquad 7+1=1+7$$

These statements are examples of *real propositions*.

On the other hand, certain other mathematical statements, like

$$a+1=1+a$$

where a represents *any integer*, are not verifiable in this way, because it is impossible to test all possible integers in the equation. To avoid the difficulty—that is, to get a "finite" rather than an "infinite" proof—this equation must be thought of as a purely formal statement, and if it is to be verified at all it must be verified by formal argument without regard to the meaning of the statement. Statements of this second type are examples of *ideal propositions*, and the formal arguments necessary to establish them are

[13] In fairness, it should be noted that later in Bell's book he again refers to Hilbert, putting concrete emphasis on recognized parts of Hilbert's program.

For a viewpoint on "meaningless marks" similar to that of the present writer, see Black, Max, *The Nature of Mathematics; a Critical Survey*. London, Kegan Paul, and New York, Harcourt Brace and Co., 1933, pp. 8–9.

to be made, according to definite rules, from previously listed axioms, with reliance now merely on our intuitive knowledge of the characteristics of marks *as marks*. The machinery necessary for this formalized procedure is supplied by the *Beweistheorie.*

The axioms of the *Beweistheorie* involve *both* fundamental logical and mathematical undefined concepts and are stated completely in symbolic form. The same naturally holds with respect to the *ideal* propositions obtained from the axioms by the use of the rules. As we have already noted, Hilbert asserts that the axioms and derived formulas are the images of the thoughts of ordinary mathematics but, *in themselves*—as configurations of marks—they have no meaning. This is in accordance with the aim of formalization in rendering infinite proofs finite.

The axioms are divided into six groups as follows: (1) axioms of implication; (2) axioms about "and" and "or"; (3) axioms of negation; (4) the "logical ϵ-axiom"; (5) axioms of equality; (6) axioms of number. Among the logical symbols used in the first four sets of axioms are the following:

$\rightarrow$ for "implies," & for "and," $\vee$ for "or," $\overline{\mathrm{A}}$ for not-A

The logical ϵ-axiom seems to be the most important single axiom and is largely responsible for making proofs finite; but its nature is too technical to explain here. Selected examples of the other axioms are the following:

I 1. $\mathrm{A}\rightarrow(\mathrm{B}\rightarrow\mathrm{A})$ ("A implies that B implies A").

II 9. $\mathrm{B}\rightarrow\mathrm{A}\vee\mathrm{B}$ (B implies A or B. This is equivalent to the primitive proposition 1.3 of the *Principia*, namely, $q\supset(p\vee q)$.)

III 12. $\overline{\overline{\mathrm{A}}}\rightarrow\mathrm{A}$ (This is the principle of the double negative, the converse of a previously noted theorem of the *Principia*).

V 14. $a=a$ (where a denotes an ordinary integer).

15. $(a=b)\rightarrow(\mathrm{A}(a)\rightarrow\mathrm{A}(b))$ (Here $\mathrm{A}(a)$ means a proposition about the number a.)

VI 16. $a'\neq 0$ (where a' denotes the next integer following a).

It is significant to note that the mathematical concepts involved in groups V and VI take for granted the system of whole numbers.

In proving new formulas from the axioms the formal rules used may be summarized as a rule of inference or implication, and a rule of substitution. The rule of implication corresponds merely to the following configuration:

$$\begin{array}{c} S \\ S \rightarrow T \\ T \end{array}$$

where each of the premises S and S→T is either an axiom or an "end-formula" proved earlier in the proof or else results from such axioms and end-formulas by substitution. The rule of substitution permits the substituting of certain groups of symbols for individual symbols in axioms and proved formulas.

According to Hilbert, the structure of mathematics—or better, the image of the structure of ordinary mathematics—is to be built up on the basis of the axioms of the *Beweistheorie* by adding new sets of axioms, and by showing that at any given stage of the process all of the axioms are consistent. The attempt to furnish a generalized method for proving consistency seems to be the second main aim of the *Beweistheorie* (the first aim being the making of finite proofs).

The method proposed for establishing consistency is based on the original axioms of negation, and is as formalized as the method of obtaining new formulas from the axioms. Briefly it may be described as follows. From the axioms of negation it follows that, given two contradictory propositions, A and $\bar{A}$,

$$(A \ \& \ \bar{A}) \rightarrow B,$$

where B is any proposition. If for B the proposition $0 \neq 0$ is substituted (by using the rule of substitution), then

$$(A \ \& \ \bar{A}) \rightarrow 0 \neq 0$$

becomes a proved formula. It asserts that wherever two contradictory propositions A and $\bar{A}$ occur together, the end-formula $0 \neq 0$ results. Hence in order to prove the consistency of any set of axioms it is only necessary to show that $0 \neq 0$ cannot result in any proof based on the axioms. Here again, Hilbert says, the task can be accomplished by reliance on our intuitive knowledge of symbols.

Hilbert admits that a great deal of work remains to be done before either of the main aims of his *Beweistheorie* can be fully realized but he is hopeful of the possibilities.

VI

It now seems possible to make the following interpretative summary of the formalists' ideal for mathematics.

According to the formalists, the ultimate foundation of mathematical knowledge is on a still more fundamental level than any logical or mathematical notions. It consists of pre-logical and pre-mathematical symbols or marks concerning which we have intuitive knowledge. Certain statements—the *real* statements, which are made about mathematical concepts by use of appropriate symbols for the concepts, can be verified intuitively; others, the *ideal* statements, cannot. But we can also verify intuitively our statements about marks as marks; consequently, in order to establish the ideal statements on as sound a basis as the real statements, it is necessary to erect a formalized structure for mathematics, which is the image of thought but which is to be developed without actual reference to thought-content.

As the specific foundation for this formalized structure the formalists propose axioms which are the images of fundamental logical and mathematical ideas, concerned, for example, with implication and ordinary integers. By following formal rules which are so chosen as to correspond with accepted processes of deductive thinking theorems are deduced from the axioms. These again are images of corresponding theorems having thought content. New axioms are introduced as a basis for continuing the process, provided at each stage the consistency of the axioms is established. The method of proving consistency is also a special formal procedure based on two of the original axioms.

Thus in the statement of axioms and theorems, in the proving of theorems, and in consistency proofs, the dependence on intuitive knowledge of symbols is clearly shown in the formalist program.

The purpose of the formalized procedure is not to make mathematics an arbitrary game with meaningless marks, but rather to render the logical structure of existing mathematics more secure, by making it more definitely objective. Hilbert gives this last point a generalized philosophical bearing in his reply to Brouwer by saying that his theory makes explicit the rules according to which our thinking proceeds, and thereby provides a basis for objective thinking in all fields, as opposed to subjective opinion and emotion.

VII

In the beginning we stated that we were especially interested in investigating the nature of the subject matter, substantive foundations, method, structure, truth, and symbolism of mathematics, from the standpoint of each of the three schools. Let us now attempt a final comparative survey with reference to these items.

First, a general survey: from the *postulational* standpoint, mathematics is a collection of deductive sciences each having its own set of postulates and undefined terms, each making free use of logic in developing its own set of theorems; from the *logical* standpoint, mathematics is a unified science which can be developed out of logical concepts, from logical principles, and by use of logical principles; from the *formalist* standpoint, the formal structure of mathematics is to be developed from certain logical and mathematical axioms, considered as images of thought, by means of formal application of the rules of deduction.

From all three standpoints the method and structure of mathematics may be called deductive, for, in each case, the program calls for assumptions and undefined terms as a starting point for the use of deductive reasoning to arrive at new conclusions. In the *postulational* school there are different starting points for the various mathematical sciences, most of which assume deductive logic without analyzing it; in the *logical* school the starting point is carried down into the primitive ideas and propositions of logic, and logic is then developed in great detail, finally merging with mathematics; in the formalist school, the most fundamental level of the foundations goes still deeper, and consists in our intuitive knowledge of pre-logical and pre-mathematical symbols, while the next higher level consists of axioms both of mathematics and logic.

From a study of both the logical and formalist schools we can gain some insight into the essential nature of the process of deductive reasoning. There is a marked resemblance between the configuration for a proof as described by Hilbert and the primitive proposition 1.11 of the *Principia*, which we have seen to be fundamental in that work in deducing one propositional function from another. Hilbert's configuration is S, S→T, *T*, while the corresponding rule of the *Principia* is $f_1, f_1 \supset f_2, f_2$ (where the assertion sign is understood to precede each of the three propositional functions). We have also noted that the process of substituting

one set of symbols for another is fundamental in the *Principia*, but it is not formulated as a definite rule so explicitly as in the formalist school. Official proofs in the *Principia* seem to be just as formalized, and the symbols used in these proofs just as meaningless *in themselves* as in the formalist school.

Although the logical and formalist schools both make the foundations of mathematics go deeper than in the postulational school, it does not necessarily follow that the deductive structures erected on the postulational foundations are less secure than the others. The soundness of a deductive structure depends upon its logical consistency, as well as on the soundness of the deductive reasoning used in developing the structure. It is not immediately apparent which of the three methods of treating consistency is preferable.

The postulational school attempts to establish the consistency of a branch of mathematics by exhibiting a concrete interpretation which satisfies all of the postulates, where the latter are viewed as abstract statements in terms of the undefined concepts. The existence of the concrete system is assumed without question, that is, intuitively, and in the last analysis it is also assumed intuitively that we know whether or not the postulates are satisfied for any particular concrete example.

The formalists, on the other hand, use almost the reverse process in establishing the consistency of their axioms; that is, the axioms are viewed as the images of concrete thoughts in the beginning, but the concrete interpretations are immediately disregarded in order to make the axioms meaningless *in themselves*. After this is done, consistency is established by formal operations on the sets of marks involved in the axioms, in accordance with our intuitive knowledge of symbols as such.

The *Principia* follows a still simpler method with respect to consistency: it simply notes that the primitive propositions are true propositions of logic. The propositions are presumably (though not explicitly) held to be consistent on this basis, just as a set of abstract postulates is held to be consistent in the postulational school if a concrete system can be found for which all of the postulates are judged to be true. The chief difference is that the primitive propositions of the *Principia* have concrete intuitive meaning in the beginning, so that the step of interpreting abstract postulates by a concrete system is missing. Surely the judgment

that the primitive propositions are true must depend on intuitive notions to just as great an extent as do corresponding judgments for a concrete system in the postulational school.

Accordingly, it seems fair to say that any judgment concerning the truth or consistency of the assumptions used in any of the three schools, depends, in the last analysis, on intuitive, unproved notions. Any absolute basis for claiming truth or consistency thus seems to be lacking.

It follows immediately that no one of the schools can claim that the theorems of mathematics, or of logic, are absolutely true. Whether the postulates are logical or mathematical, the theorems are true at most in relation to the postulates and in relation to further methods or rules of deductive procedure used, no matter whether such rules are tacitly taken for granted or are stated formally or informally.

If we compare the nature of mathematical subject-matter according to the three schools we find somewhat more variation. According to the postulational school the subject-matter may be either abstract, or concrete with intuitive significance in numerous directions; according to the logical view, the subject-matter is such as can be built up by the process of definition from the primitive concrete ideas of logic; while, according to the formalist school, the formalized structure of mathematics for purposes of rigor is deprived of subject matter other than symbols, but the symbols remain the images of concrete thought. In the formalist school the concepts used in the *real propositions* of mathematics should also be included with the subject matter; these concepts, we remember, are held to be of a direct intuitive nature like the symbols themselves.

Thus, in each of the schools, the subject matter, as well as truth or consistency, seems to be ultimately traceable to intuitive concepts. But in no case do we find any limitations of the concepts of mathematics to notions of "number and space."

Finally, we can compare the function of symbolism in the specific programs of the three schools. In the postulational school the symbols used for the undefined terms seem to be very convenient, but non-essential so far as the investigation of sets of postulates is concerned. In the official portion of the *Principia*, meaningful symbolic language is used throughout as an indispensable aid to the precise development of the subject and to the

giving of proofs which apparently are just as formalized as those of the formalist school. In the formalist school symbols play a still more fundamental part, for they furnish a key both to formalized proofs and to proofs of consistency.

We are now in a position to draw certain general conclusions concerning the nature of mathematics, when viewed in the light of modern investigations into the foundations. We shall not refer again to the nature of mathematical symbolism, because we do not seem justified in generalizing concerning the use of symbolism in mathematics as a whole, simply on the basis of its use in the foundations.

First, the *subject matter* of mathematics is not restricted to ideas of "number and space." From the modern point of view the subject matter may include logic, abstract sciences, and a wide range of concrete interpretations. The ultimate origin of the subject-matter seems to be in intuitive ideas.

Second, a starting point consisting of assumptions and undefined terms is necessary for the development of any mathematical structure. No matter whether this starting point is explicitly formulated in logic, beyond logic, or prior to logic, any judgment concerning the truth of the assumptions (or concerning the degree to which they are satisfied for a given concrete interpretation) seems to depend, in the last analysis, on intuitive considerations.

Third, the method by which a mathematical structure is developed is the method of deductive reasoning used in obtaining theorems from the fundamental assumptions, and the corresponding process of defining new concepts with ultimate dependence on the original undefined concepts.

Fourth, the soundness of a mathematical structure of thought depends on the soundness of the deductive reasoning used in developing it, and on the consistency of the original assumptions. So far no *absolute* basis has been established for judging whether these requirements are met.

Fifth, the theorems of mathematics are not *absolute truths*. They are true at most in relation to the postulates from which they were deduced, and the methods of reasoning by which they were deduced.

The contrast of these conclusions with the traditional view of mathematics is striking, and needs no further comment. But it is not safe to claim that we have presented any final picture of the

nature of mathematics; for the fundamental concepts and methods of mathematics are perpetually in a state of evolution.

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The Use of Original Exercises in Geometry

By MABEL SYKES
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RECENT geometry texts contain a very large number of original exercises; so many that the question of how to handle them is often serious. This is especially true for teachers who believe that it is necessary to take everything in the text in use and take it in the order given. Many of these are young teachers or teachers whose major is not mathematics, teachers who for one reason or another do not trust themselves to cut or to rearrange the material in the text, teachers who do not seem to appreciate that they are teaching geometry, not teaching a book.

GENERAL PRINCIPLES

Certain principles should govern the use of exercises. These will be discussed separately.

I. It was never intended that all of the exercises in any book be given to any one class.

II. In only rare cases are the exercises in the book so arranged that they must be assigned in the order given.

III. Exercises should not be assigned in a haphazard way as mere busy work but should be carefully selected to illustrate the application of some theorem or group of theorems or some method of proof.

IV. No exercise should ever be assigned that the teacher has not worked out to his own satisfaction and found reasonably easy.

I. *It was never intended that all of the exercises in any book be given to any one class.*

To the exercises in the text should be added in some cases a large number of corollaries and unnumbered theorems. These theorems and corollaries are an integral part of the work, inasmuch as it is expected that they will be referred to in proofs that follow. They should be proved. They are particularly annoying to pupils as in many books the proofs of the regular theorems are given more or less in full, while the proofs of the corollaries and extra theorems may not be even suggested. It is evident that they should be taken

into account in estimating the amount of required work. Corollaries and unnumbered theorems act largely as camouflage to reduce the number of regular theorems, that is, reduce the apparent amount of required work.

Teachers who believe that it is necessary to take all of the exercises in the book in use, must find themselves in something of a dilemma. A little computation will show that they have set themselves an impossible task.

The exercises were counted in five texts of recent date more or less widely used and were found to range from 1440 to over 2100.

Hardly more than 36 weeks can be counted on for regular class work out of a year of 40 weeks, because one must take into account the holidays scattered throughout the year and the time necessary at the end of each semester for the closing of records.

To cover the amount of original work given in the texts referred to above in 36 weeks, it is necessary to give in class an average of from 40 to nearly 60 exercises a week. This gives an average of from 8 to 12 a day and allows 5 minutes or less for each exercise in a 45 minute lesson. Much of the value of exercise work, especially to the weaker pupils, is in the discussion that follows the report on the exercise. There is, in five minutes, very little time for profitable discussion, while no time has been allowed for reports on, and discussion of, the regular theorems. It is evidently impossible to cover everything in these texts as classes are ordinarily conducted.

If the class is conducted on some variation of the individual plan much work outside of class must be accepted because class time must be allowed for oral reports from, and explanations to, individual pupils. The only way to be sure that a pupil, or group of pupils, really understand an exercise or theorem, is by some oral report or discussion of the same. The amount of outside work required under these circumstances for any pupil to cover all of the text, is such that the burden can be successfully carried only by star pupils. The rest must either copy from their neighbors or fall by the wayside. It is evidently impossible for all of the class to cover all of the text intelligently in this way.

While it was never intended that all of the exercises in any one book be given to any one class the large number of exercises in recent texts serves certain well defined and useful purposes.

The time is long passed when a geometry teacher can assign lessons from a text day after day without preparation in planning

his work and get results worthy the name of geometry. Even with the best of texts he cannot do so today. This is partly, at least, due to the wide range of individual differences in the pupils in our classes and to the large number of pupils for whom the work as given 15 or even 10 years ago is not fitted. One problem of text-book makers is to reduce in some degree the amount of outside work required of individual teachers.

It is easier for most teachers to omit exercises if the text contains too many than it is to supplement those in the text from outside sources. A large number of exercises in the text in use, does away with the necessity of keeping on hand several extra books, of consulting them regularly and of getting the exercises selected into the hands of pupils so that each member of the class will have a correct copy that will not be readily lost.

A text containing a large number of exercises can be readily adapted to different types of classes. The pressing problem of individual differences is largely met in our schools in two ways, by ability grouping and by differentiated assignments without ability grouping. When ability grouping is used, it is not often that different texts are used for classes of different grades of ability. Whatever is the method of meeting the problem of individual differences, it is evident that different classes and different pupils need different work. The weaker classes and the weaker pupils need a large number of easy exercises designed to give them an appreciation of the meaning and application of the fundamental theorems and methods. A good deal of easy drill is essential. Such pupils do not need and will not profit by the harder work. The better pupils and the better classes on the other hand do not need so much of the easy drill but do need enough of the harder exercises to make the work seem worth while and to give them something of the joy of achievement.

Unless an immense amount of outside work is to be done by the teacher, such a careful selection of exercises can be satisfactorily obtained only when the text in use contains a very large number of exercises many of which are very easy.

Under the best conditions classes differ and the point of view of any one teacher differs from year to year. With a large number of exercises in the text in use the work can be varied at different times and for different classes. This serves to keep the work fresh.

The teacher's own problem of keeping supplementary exercises so that the work need not be duplicated from year to year is not a simple one. Such exercises may, of course, be written on cards and kept. Each card should show the figure used, the hypothesis and the conclusion. If, however, there is no way of keeping these cards, they soon accumulate in disorder and are lost. To prevent this a card index file with a double set of guides is recommended. It is very useful even to teachers who do not need a large supply of outside exercises as some sudden turn in the lesson often makes an exercise illustrating some particular point immediately desirable. It is quicker to look over a package of cards arranged to illustrate some definite theorem than it is to turn the pages of a book.

II. *In only rare cases are the exercises in the book so arranged that they must be assigned in the order given.*

It is evident, of course, that if exercises are graded and grouped so as to be suitable for weaker, better and star pupils respectively, this grouping should be respected.

Occasionally among the harder exercises a series occurs, the point of which is lost unless they are assigned in the order given. But generally this is not true and the order may be disregarded. Indeed a change in the order of assigning exercises is often desirable. If, for example, the teacher wishes to give extra drill on some particular point, exercises illustrating this point should be selected irrespective of order.

III. *Exercises should not be assigned in a haphazard way as mere busy work, but should be carefully selected to illustrate some theorem, or group of theorems, or some method of proof.*

If pupils are to obtain a clear and lasting impression of geometry that will make their work worth while, unity and emphasis in the presentation of that work are necessary. These are best obtained by some variation of the unit or topic method of teaching. The work should, therefore, be carefully planned with the idea of unity and emphasis in mind and the exercises should be so selected as to bring out the application, first of the more important and fundamental theorems, and later of the minor ones, if the time permits and the class is able to take them. This cannot be done by carelessly assigning any exercise that comes along.

If exercises are studied carefully it will be seen that either they contribute directly to the unity of the work at the place in which they are given or are best suited for review work, in that they depend on something previously studied but not directly included in the topic under discussion. They should be used for the purpose for which they are best fitted. Such review exercises are often found at the bottom of pages after the theorems. They have been called "page fillers," because the author found it necessary to insert one or two exercises to bring the page to the required length.

IV. *No exercise should ever be assigned that the teacher has not worked out to his own satisfaction and found reasonably easy.*

If an exercise is too hard for the teacher it is too hard for his class. An exercise may have been given in the text because of the application of some special theorem that makes it easier than would at first appear. This easier method may be discovered by some pupil. At the same time no teacher should ever allow himself to be placed in the unfortunate position of appearing before his class unable to do the assigned work.

Even if an exercise is not very difficult it may not fit into the unity of the topic under discussion. All exercises should be carefully worked out by the teacher in advance and a careful selection made with the idea of unity in mind. This is an entirely different thing from assigning exercises in the order given in the text and then working them out after the assignment is made.

Suggestions for Easy Exercises

It is especially important that a large number of easy exercises be given to weaker classes and pupils at the beginning of each new topic, and after each fundamental theorem or group of theorems. Below are some suggestions showing what is meant. The topic chosen is Circles and Related Lines. Two groups of exercises are included.

I. The object of the exercises in this group is to familiarize the pupil with the meaning and application of the theorems studied. They are best fitted for oral work. If the theorems to be illustrated are those concerning the relations between equal chords, arcs and central angles the following serve as samples:

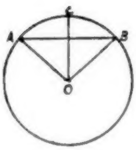

FIG. 1

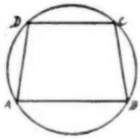

FIG. 2

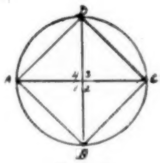

FIG. 3

1. If in Fig. 1, O is the center of the circle, and OC bisects $\angle O$, what theorem would be used to prove arc AC = arc CB.? Quote in full.

2. What do you know in Fig. 2 if chord AD = chord BC? Quote the theorem used.

3. What do you know in Fig. 2 if arc AD = arc BC? Quote the theorem used.

4. If in Fig. 3 $ABCD$ is a square, why are the arcs AB, BC, CD, and AD equal? Quote the reasons in full.

5. If in Fig. 3 AC and BD are $\perp$ diameters, what theorem would be used to prove (1) the chords AB, BC, CD, and DA equal? (2) the arcs AB, BC, CD, and DA equal? Quote in full.

Similar questions should be asked after the theorem concerning the perpendicular from the center of a circle to a chord and after the theorems concerning tangents. If the book does not contain work of this type, such exercises can easily be made up by the teacher by turning to any figure, or figures in the text that illustrate the theorems studied. Such figures should be different and somewhat more complicated than that used to prove the theorem itself. The number of such exercises to be given depends upon circumstances.

II. Exercises like the following may then be given. These may be oral or written as appears necessary or desirable:

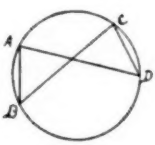

FIG. 4

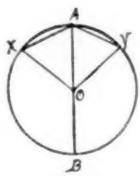

FIG. 5

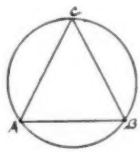

FIG. 6

6. In Fig. 4 if chord AB = chord CD, prove that arc BC and arc AD are equal, and that chord BC and chord AD are equal.

7. If in Fig. 4 chord BC = chord AD, prove that chord AB and chord DC are equal.

8. If in Fig. 5, O is the center of the circle and OA bisects $\angle XOY$, prove that chord AX and chord AY are equal and that arc BX and arc BY are equal.

9. If in Fig. 6, $\angle A = \angle B$, prove without measurement of angles that arc AC = arc CB.

After the study of the topic is well under way, harder exercises involving several of the important theorems can be given to the better pupils or to the better class.

Special Methods of Assigning Exercises

At the close of a topic or for review work it is often interesting to post somewhere in the room a list of suitable exercises from which each member of the class is to choose one on which he is to report orally. Each pupil signs his name to the exercise that he has chosen. However, lest the more aggressive pupils sign up immediately for the easy ones, the lists may be divided into C, B and A groups and pupils told from which list to choose. Pupils learn a good deal when the reports on these exercises are given, especially if the report is followed by some questioning and a lively discussion.

If is often interesting to make more or less of an indefinite assignment especially for better pupils, as, for example, any three of four exercises from certain pages. Pupils browse around when the assignment is made in this way and often bring in some interesting results. To prevent a run on some special problem it is sometimes necessary to refuse to receive any more reports on that particular exercise. The better pupils may sometimes be required to report orally on any exercise on which no other pupil has reported. The choice in this case should be limited to exercises on certain pages.

For further discussion of the subject see the author's Monograph on The Teaching of Geometry, by Rand, McNally & Co.

There is Fusion in the Air

By Aaron Freilich
Bushwick High School, New York City

Despite the great strides education has made through modern experimentation, many laymen and even teachers not only refrain from supporting further experiments, but actually frown upon the conclusions deduced from them. Frankly, too much censure should not be heaped upon such dissenters, for have they ever seen the original report of an experiment to admit failure? Are not all experiments successes, that is, at least until after the results are written up and published? There is no doubt but that the ends of education would be served better if experimenters would not color their reports merely for college credit, publication fame, or personal gain.

As regards the experiments dealing with fusion or correlation of work, all I have seen give results that are favorable and encouraging. The validity and reliability of these results are strengthened by the fact that of all modern educational trends, I doubt if any one of them has won as much approval and as little criticism from outstanding leaders, as has the idea of fusion. Because of this, one should be willing to accept the very favorable reports of these experiments.

The idea of fusion or maximum correlation of work is old, has persisted, and is now stronger and more widespread than ever. In fact, the fusion idea has made strong inroads into several subjects in both the college and high school.

In the Colleges

Science. The New York Sun, on May 31, 1934, reported as follows:

"Adoption by the faculty of Columbia College of a new two-year elective course in the sciences, which constitutes another major departure from the traditions of higher education, is announced by Dean Herbert E. Hawkes.

"The sciences, long treated as separate fields, have been organized as a unified sphere of study, dominated by great scientific ideas rather than by technics, it is explained. The purpose of the

course, which goes into effect next fall, is to 'afford a wider view of scientific subject matter than is possible by a study of only one or two sciences and to produce a broader outlook in the student not only upon the several sciences but upon those problems and ideas which the sciences share with each other and with the other domains of contemporary thought.'

"The course, which will meet the science requirement for the degree of bachelor of arts, becomes a part of the college curriculum after fifteen years of faculty effort. In its aim and its importance it is compared at Columbia with the course on contemporary civilization which was adopted soon after the close of the world war."

Social Studies. See reference to this in the above quotation from the New York Sun.

Mathematics. Many colleges now give in the first year either a fusion course or a course which treats several branches of mathematics. These courses are called by various names such as Freshman Mathematics, Fundamentals of Mathematics, etc., and if the work is not unified it is given in tandem order.

In the High Schools

In General. The May, 1934, issue of High Points contains an article by Mr. William A. Clarke, Principal of John Adams High School, New York City, entitled "What Does the State Survey Report Mean to Us?" The following paragraph, quoted verbatim from this article, has a direct bearing on our discussion. In it one gets a glimpse of the future high school curriculum.

"There is need of an integrated curriculum not only in subjects begun in the junior high school and continued in the senior high school, such as the foreign languages, mathematics, and commercial studies, but also within any particular school department or subject class. In too many cases, the old notion persists of intense cultivation of a very limited field, in each subject or class, with little or no contact beyond that field. We must break down more and more the artificial barriers that exist between subject and subject and insist upon integrated studies with the traditional subject fields serving as the cores of larger spheres of learning that will engage the pupil's attention and activity. Opportunities for such correlation or integration are many in the traditional 'humanities'—the social studies, foreign languages and English. It is only

through integration in the fullest sense of the word—educational philosophy, guidance, curriculum, methods—that we can build up the proper organization of material and of learning procedures that will satisfy the individual pupil's capacity and needs."

Science. Nearly all high schools now give a preliminary composite course in science, usually called General Science, in which the subject matter is taken from physics, biology, chemistry, etc. This course is often a pre-requisite for all the separate science courses.

Social Studies. The New York City chairmen of the Social Science departments are now discussing a proposal, advanced by some of its members, which advocates an integration of all the social studies into a four year sequence.

Mathematics. In 1923 the National Committee on Mathematical Requirements reported as follows:

"In recent years there has developed among many progressive teachers a very significant movement away from the older rigid division into 'subjects' such as arithmetic, algebra, and geometry, each of which shall be 'completed' before another is begun, and toward a rational breaking down of the barriers separating these subjects, in the interest of an organization of subject-matter that will offer a psychologically and pedagogically more effective approach to the study of mathematics."

Since this epochal report first appeared important progress has been made along the lines of fusing, i.e., coordinating the subject matter in mathematics. That the fusion idea has exerted a great influence over the country is evidenced by the fact that every elementary algebra text-book published in the last five years has in it some intuitive geometry as well as a unit on numerical trigonometry, just as the plane geometry texts of recent years include more and more algebraic applications, the unit in numerical trigonometry, and some applications to solid geometry.

In the Mathematics Teacher of October, 1933, Dr. E. R. Breslich of Chicago says, "It seems therefore that from the standpoint of education in general as well as from the standpoint of mathematics the need for the correlation of the various mathematical subjects was never more apparent than at the present time. Better mathematical training will be obtained from the study of mathematics than from the successive study of arithmetic first, then algebra and then geometry." Again he says, "However, the measured results

that are available seem to favor the plan. In general the investigations that have been made establish the fact that the results obtained with combined courses are at least as good as those obtained with the separate courses when measured with tests in algebra and geometry. However, when mathematical power tests are used, as for example tests of ability to use and apply mathematics and tests of functional thinking, the results obtained with the combined courses indicate superiority."

At a meeting in Cambridge, Mass., on March 10, 1934, the Association of Teachers of Mathematics in New England passed the following two recommendations concerning advanced mathematics for communication to the Commission on Mathematics of the College Entrance Examination Board.

"A full year course consisting of trigonometry, solid geometry, and advanced algebra is to be preferred to the present practice of offering half-year courses in only two of these three subjects.

The College Entrance Examination Board is requested to consider the desirability of replacing the present examinations in advanced mathematics, by a single examination on an undivided year course in advanced mathematics, the paper to be so constructed that questions must be answered in each subject while permitting sufficient option to allow teachers some latitude in selecting material for instruction."

It is my understanding that the College Entrance Examination Board has already agreed upon and decided to give comprehensive examinations in mathematics, thus eliminating the separate examinations.

The Standing Committee for the New York City Association of Chairmen of Mathematics Departments has just completed a syllabus correlating elementary algebra and plane geometry for the ninth and tenth years. The New York State Board of Regents now has in operation both a syllabus and a Regents examination for a full year course in fusion mathematics, consisting of intermediate algebra and plane trigonometry. For the latter course textbooks are now available, the authors of which claim success in fusing these two subjects.

It is worth while noting some of the advantages claimed for fusion mathematics.

1. It is possible to make the function concept the unifying and underlying idea of all mathematics instruction.

2. There is a greater exposure to the topics in the several subjects that are fused.

3. Each subject is strengthened by the other, when taken together.

4. There is a wider field for applications.

5. It gives more power to the learner since he has the liberty to use either one or all of the subjects to solve his problem.

6. It leads to more thorough understanding and better retention.

7. It will result in greater numbers benefiting by the study of mathematics since pupils who want more than two years of mathematics will have to take another full year instead of a half year course as at present.

8. It reduces the number of Regents and College Entrance Board examinations.

9. It eliminates much of the review and repetition which are necessary in the old water-tight compartment division of subject matter.

10. European experience with fusion on a large scale has been entirely satisfactory.

That fusion mathematics is here to stay follows from the inherent advantages in such a course. There is no question but that further progress in the secondary mathematics field lies in the direction of fusion mathematics.

The Place of Mathematics in the Curriculum of the Progressive School

By Charles Russell Atherton
Teachers College, Columbia University, New York, New York

Two questions confront the curriculum committee of any progressive institution when the course of study in any particular subject matter field is under consideration. First:—What is the claim of the particular subject upon a place in the program? Second:—How can the administrative difficulties involved in the offering of courses in the subject be overcome?

These are the questions which I am going to attempt to answer here but before going farther it is perhaps necessary to make a few definitions of terms which will crop up from time to time during my talk.

By the *progressive or "integrated" programme* I mean the program which, in its entirety, attempts to show the interrelationship between the great fields of human knowledge—namely, Mathematics and the Natural Sciences, the Social Sciences, English and the Arts, and Philosophy, Religion, and Ethics.

By *mathematics* I shall mean not only the formal mathematics that has for too long dominated our institutions of secondary and higher education but also the study of relationships of any quantitative nature whatsoever, and the mathematical mode of thought.

By a *unit course* I shall mean a course which deals with several topics generally classified under the same subject matter title, each topic being independent of the other and admitting the opportunity for a student to take any number of desired units without the necessity of taking the complete course.

That mathematics has a direct and a close relationship to the field of the natural sciences is beyond cavil. Particularly in physics and chemistry and at a later date in biology mathematics has been the most powerful tool of scientific investigation. It is of such importance that it is safe to say that progress in the development of scientific research is limited chiefly by the limitations of extent in our knowledge of mathematics.

Of course I do not claim that every student in the field of the natural sciences should take a great deal of pure or applied mathe-

mathics but it is impossible to understand why a person who finds himself particularly interested in this field will allow himself to be handicapped in his later work by a lack of fundamental mathematical training. This point should be made clear to the science majors in any college and courses should be required of them as rigorous and thorough as those required of majors in the field of mathematics for at least the first two years of the course, or, up to and including the differential and integral calculus. This will give these students the background necessary for more or less advanced work in the sciences and open the door to further study of advanced mathematics when and if their progress becomes hindered due to a lack of background in that field.

In the social sciences the problem appears to center in two outstanding fields, namely, geography and economics. Geography in the modern sense is a course in the humanities and the mathematics of the subject is simple and interesting unless the field of navigation is included with geography. For this reason I am in favor of grouping navigation and astronomy together in the field of natural science.

Economics, however, has a different aspect and the ability to predict trends by mathematical methods is increasing daily in its importance in this field. Some of the important implications of mathematics in connection with economics include the ability to construct and interpret several different types of graphs, a knowledge of seasonal and cyclical variation, the construction and interpretation of index numbers and a knowledge of their limitations, the use of interpolation and extrapolation for predictive purposes, and a knowledge of the mathematical bases of the quantitative theory of money. It is also important that students in this field be aware of the limitations of the quantitative methods used in economic investigations.

In the field of Philosophy mathematics has a recognised niche of its own. Here the power of the mathematical methods of thought, the use of mathematics as an exponent of postulational reasoning and deductive thinking, is held in high regard. Moreover, in the various branches of geometry we find the only applications of logic in pure form which can be understood successfully by the immature student.

In the arts and English it is conceivable that the student should disregard mathematics entirely as an actual tool but, historically,

art and mathematics are inextricably interlaced. The mathematics of sound is important in music, Greek statuary and architecture is based on the principles of dynamic symmetry, the paintings of the great masters are conceived on mathematical principles of perspective and proportion, and even the mixing of colors falls back upon mathematical laws. It is well known that the great masterpieces of Leonardo da Vinci were the result, to a large degree, of his mathematical method. From the point of view of the artist or sculptor these statements may be met with the reply that, in many cases, the mathematics applied was to a large extent intuitive and not the result of a study of the subject. Nevertheless the fact of its use is constantly apparent and it is this fact which is important rather than the method which brought it about.

Up to this point I have made no mention of the study of mathematics for its own sake nor do I advocate it for those who are adverse to it. There are always those students, however, who take keen delight in the use of their powers of reasoning in abstract situations, who appreciate the mathematical elegance of some of the historic proofs which established the place of mathematics in the past and indexed its place for the future, and who are always on the watch for new and interesting problems which they may themselves attack. These students have a right to real courses and inspiring instruction which is at least on a par with the offering in the other fields.

It may be trite to recall the fact that the main plank in the platform of progressive education is academic freedom and liberty on the part of the students but we must bear in mind that "progressive" is merely an adjective which modifies the more important term "education." If this fact is forgotten, liberty on the part of the students will give way to academic license and the progressive ideal will follow other ideally conceived but inadequately administered educational movements into the limbo of discarded devices. Standards of some sort must be set and maintained but these standards must show a distinct improvement over the traditional point-credit standards in the matter of confronting the students with the problems of a changing society.

It is doubtful whether absolute academic freedom, with its accompanying implication that students of college age are completely equipped to select for themselves the courses which will

best satisfy the objectives which we as educators have set for their education, is justified. It seems patent that careful advice to students and a complete understanding between students and their advisers is the salient point of the question. Certain standards, flexible though they may be, must confront the student on his entrance to college and the courses involved in attaining such standards must be directly organized to achieve the general aims and educational ideals of the progressive school. We cannot expect that any individual or group of students can, by individual investigation or research, determine and outline a course satisfying the standards set up for instruction more satisfactorily than those whose training and experience has been designed to equip them with this ability.

From rigid standards the change must be towards more relaxed standards but some measure of academic accomplishment must be maintained if the cause of progressive education is worthy, and I feel certain that it is, of achievement.

Bearing this fact in mind we are ready to attack the administrative difficulty involved in offering courses in the field of mathematics. The most regrettable fact in this regard is that so many students entering college dislike mathematics and can be persuaded to take it only with the greatest difficulty if at all. It is impossible that the subject itself can be anathema to so many students so that the aversion must be placed at some other source, probably the type of instruction which is found on the high school level. If this unreasonable animosity toward such an important and interesting field of learning is not to continue throughout college the quality of instruction must be improved and made more interesting. Once this is done mathematics will gain in prestige and become a desired rather than avoided subject. Hence, the first administrative problem in mathematics, as in all other fields, is the improvement of the quality of teaching on the college level.

In the next place the subject matter must be relevant to the field of the pupil's major interest and hence, beyond a certain point, there must be a sharp differentiation of subject matter. In addition pure mathematics must give way to experimental mathematics and concrete illustration and experiment must become prominent in the mathematics classroom. Here again the answer is a teacher well versed in the ramifications of his subject and cognizant of the needs and interests of his pupils.

Under the very ideal of the progressive programme which calls for a background in all the great fields of human knowledge it is inconceivable that mathematics should not play a part. The material involved in this part should be required of all students and should be, in essence, a survey course dealing largely with elementary types of work, and calling to a large extend on the simple but dramatic content of the historical background of the subject. A knowledge of the growth of our number system, ancient methods of computation, the mathematical mysticism of the Hindu astrologers, and many other things of this sort are informational and highly interesting and will find an important place in the student's fund of knowledge.

For those pupils who plan to major in mathematics there is little argument as to what is desirable in the beginning but the material given to these pupils must take its beginning, not in formalized instruction and the use of ordinary textbooks, but in concrete experience with things mathematical and an ardent desire to go farther into the subject. Early in the course it should be pointed out that it would be impossible for any individual to master all that is known of mathematics but that there must be specialization within the field. Certain courses should be indicated as necessary to specialization and the pupil should receive careful advice as to the line in which he plans to specialize.

For students of subjects other than mathematics I would have one regularly constructed course in connection with each of the four great fields of learning, these courses to be arranged on a unit plan to cover material deemed necessary for efficiency and educational background by the instructor in mathematics working in conjunction with the instructors in each of the other fields.

To illustrate from the field of the Social Sciences, a regularly scheduled course of mathematics as applied to the social sciences should parallel the courses in social science, the instructor in mathematics to act as a collaborator on all topics deserving of a mathematical interpretation. The necessity of such interpretation should be based on the judgment of the instructors rather than that of the pupils and if proper coordination is effected the motivation will follow naturally and easily from the interest of the student in his major subject and faith in the sincerity of his adviser.

The units in such a course might be as follows:

(1) *Graphs and Graphic Methods.* Types of graphs, the value of each type for representation of data, the faults of each, methods of construction, interpretation, standards of graphic construction, limitations of graphic methods, etc. . . .

(2) *Statistics and Statistical Method.* (Best illustrated from a discussion of the distribution of wealth) Central tendencies, deviation, normalcy, skewness, quartiles, percentiles, etc. . . .

(3) *Risk, Risk-bearing, Insurance, and Insecurity Problems.* The use of the laws of probability in business, probable error in computations, estimation of reliability, interpolation, extrapolation, etc. . . .

(4) *Misinterpretation of Statistical Data.* Errors of this type affect social science problems. What are the limitations of quantitative work in the field of social science? etc. . . .

(5) *Social Change, Economic and Social Theory.* Laissez-faire is based on the mathematical rationalism of the 17th and 18th centuries. Mathematics as a scientific method has greatly influenced the field of economics. The essential character of mathmatical analysis should be indicated to the student, etc. . . .

(6) *Raising the Standard of Living.* The analysis of living costs is based on index numbers which are mathematically constructed. Budgets and budget data are involved in their construction. What are the limitations imposed on the use of these numbers? Problems of production and consumption, etc. . . .

(7) *The Business Cycle.* Use of index numbers, correlation, estimation of seasonal, cyclical, and secular change, use of the graph to represent the cycle and its changes. Methods of prediction other than graphic ones are equally useful, etc. . . .

(8) *Problems of Money, Banking, and Foreign Exchange.*

It is easily seen that the above units are important both from the standpoint of mathematics and economics. The student may be allowed to take any unit or group of units or the entire course. In a course of this type the units should not be definitely fixed in their sequence but should correlate closely with the order followed in the course in economics.

Too often a course involving mathematics slides over any reference to the subject with a resulting depreciation in the value of the

course itself. This should not be the case. One of the assumptions of the progressive movement involves the individuality of the pupil and I must admit that we should not bring too much pressure to bear on the individual in order to make him take any one special course. Nevertheless, I also feel that at the progressive type of college only the better type of student should be admitted during the experimental years and it seems certain that, to a large degree, such students may be relied upon to accept well-meant advice in regard to the courses which they take. The mathematics course outlined above, if carefully taught, should present no great difficulty to the good student whose major interest is in the field of the social sciences.

It is not the purpose of instructors in mathematics to force what seems to many to be an undesirable subject upon the students. We make no brief for pushing the student farther into the realm of Mathesis than he wishes to penetrate but we do feel that the actual and practical applications of mathematical knowledge which permeate all the fields of learning should be brought to the attention of the student, that the curtain should be pushed back and the beauty of the subject exposed to the light, and we feel that since mathematics *is* one of the great fields of human knowledge, that since it is according to Bell "The Queen of the Sciences," it merits a prominent place in the curriculum of the progressive school.

We decry the thoughtless and inefficient teaching of mathematics which has made it disliked by so many students but we cannot blame the subject itself for this situation. We feel that despite this difficulty there is much that will be of real benefit to any student even in the field of elementary mathematics. We make no plea for the perpetuation of any subject or topic which does not fit in with the general ideals of the progressive curriculum and serve as additional material for broadening and deepening the cultural background of the individual, but we feel sincerely that in the evolution of the integrated programme mathematics must play an important part. Authorities in every field may be quoted as to the value of mathematics. Let us help the pupils in their achievement of a well-rounded education by exposing the constant recurrences of mathematics and the mathematical forms of thinking which occur in their studies and in their daily life. Let us make mathematics palatable and interesting to the average student whatever

his major interest. The feeling of opposition to the subject must be eliminated and it lies with a forward-looking and progressive type of school to achieve this end, since such achievement is in accordance with the very ideals which brought the progressive institutions into existence.

Program for the Sixteenth Annual Meeting of the National Council of Teachers of Mathematics

Chelsea Hotel, Atlantic City, N. J.

February 22 and 23, 1934

11:00 A.M. Business Meeting of the Board of Directors.

12:30 P.M. Luncheon Meeting of the Board of Directors.

2:00 P.M. Business Meeting of the Council.
(Open to all members)
1. Reports of standing committees with a discussion of educational policies.
2. Election of officers.

6:00 P.M. Dinner for directors and official delegates of branches and affiliated organizations.

8:00 P.M. First Program Meeting.
1. Address of Welcome
Dr. Frank J. McMackin, Jersey City, N. J.
2. Response.
Wm. Betz, Rochester, N. Y.
3. *The Psychology of the Transfer of Training.*
Dr. P. T. Orata, Ohio State University.
4. *Methods of Teaching for the Maximum Amount of Transfer.*
Harold P. Fawcett, University High School, Columbus, Ohio.

Saturday, February 23

9:30 A.M. Second Program Meeting, Vice President Allan R. Congdon presiding. Symposium on the teaching of arithmetic—Directed by Mr. C. L. Thiele of Detroit.

12:00 P.M. Joint luncheon meeting of the new and old boards of directors.

2:00 P.M. Third Program Meeting.
General Topic—The Subject We Teach.
Arranged by first vice president R. D. Beatley.
1. *The Geometry of Inversion.*
Professor Roger Johnson, Brooklyn College, Brooklyn, N. Y.
2. *Graphical Methods in Mathematics.*
Professor George W. Mullins, Barnard College, Columbia University, New York, N. Y.

6:30 P.M. Annual Banquet. The banquet program will emphasize the history of mathematics.
Address on the history of mathematics.
(Speaker to be chosen)

The Tree of Knowledge

By W. D. Reeve

Teachers College, Columbia University
New York, New York

In the Hall of Science of the Century of Progress Exhibition last year in Chicago there was represented on one of the walls the "Tree of Knowledge," a photograph[1] of which we present (by permission) on page four of this issue of *The Mathematics Teacher*. As can be seen from this picture mathematics furnishes the central root and vitalizing energy for the basic sciences such as astronomy, botany, chemistry, geology, and physics and, together with them, furnishes strong support for the applied sciences on the higher branches such as social studies like economics and sociology and engineering of various kinds.

Importance of Mathematics

The fact that mathematics is so important is not surprising to one who is properly informed as to the contributions which it has made to the other great fields of knowledge, but many people including some of our educators are still unaware of the strategic place which mathematics really occupies in world affairs today. It should be the business of those of us who are interested primarily in the subject to help to make clear just where and how mathematics can be of real service to the other great branches of learning and what can be done to secure these services by a better teaching of mathematics in the schools.

The Universality of Mathematics

In this complex civilization which we are now entering a knowledge of mathematics is becoming increasingly important.[2] This does not mean that everyone should be trained to be a mathematician, but it does mean that every well educated citizen in America should know a reasonable amount of mathematics and also that he should be trained to use it in an intelligent manner.

[1] Reprints of this photograph may be secured for 5¢ postpaid by writing to *The Mathematics Teacher*, 525 West 120th Street, New York City.

[2] Reeve, W. D. The Universality of Mathematics. *The Mathematics Teacher*, February, 1930.

Abstracts of Recent Articles on Mathematical Topics in Other Periodicals

By NATHAN LAZAR

Alexander Hamilton High School, Brooklyn, New York

Arithmetic

1. Andrews, F. Emerson. *An excursion in numbers*. The Atlantic Monthly. 154:459–466. October 1934.

A very readable and interesting account of the development of the decimal system. Its disadvantages are enumerated and the unquestionable superiority of the *duodecimal* system pointed out. The author fervently advocates the introduction of the duodecimal system, but he is not blind to the obstacles to be met and the adjustments that will have to be made in case of its adoption. "The present generation would have a most awkward time, chiefly in unlearning the old multiplication tables; but children for all future generations would find mathematics made vastly easier by the present sacrifice."

2. Walters, Margaret R. *How to study arithmetic*. School Science and Mathematics. 34:848–852. November 1934.

A summary in outline form of a Master's thesis, University of California, June 1933. The difficulties encountered in studying arithmetic and the suggestions for overcoming them are outlined under the preceeding eight main headings, with 10-20 subheadings under each

a. How to learn the terminology of arithmetic,
b. How to learn the fundamental processes,
c. How to study fractions and decimals,
d. How to study mensuration,
e. How to do arithmetic problems,
f. How to check arithmetic work,
g. How to acquire accuracy and speed, and
h. How to use labor saving devices.

The hints given under each of the headings seem to be sound psychologically. Without seeing the complete thesis it is impossible, however, to determine on what basis the recommendations were made,—whether they were based on experimental data or whether they were merely derived from the accepted laws of learning.

Algebra

1. Lazar, Nathan. *The advantages of teaching the solution of verbal problems by the multiple-equation method*. High Points. November 1934. pp. 28–37.

A paper read on February 17, 1934, at the annual open luncheon and meeting of the Mathematics Chairmen's Association of New York City. The thesis is threefold:

I. It is possible to teach the solution of problems by using *many* symbols and *many* equations from the very beginning of the course.

II. There are certain objections to the traditional system, which are met by the multiple-equation method:

1. The beginner, who is brought up in the usual approach, gets the erroneous impression that algebraic problems generally involve but one equation and but one central thought.

2. The one unknown method permits three and sometimes four different ways of solving even so simple a problem as: "Divide 30 into two parts so that the smaller is four less than the larger."
3. The traditional method of solving problems does not give the pupil ample opportunity to learn the true appreciation of the value and power of symbolic representation.
4. There is a scarcity of problems in one unknown. The overwhelming majority of problems in text books and on uniform state and entrace examinations deal with more than one unknown.
5. The use of boxes in the solution of certain types of problems is an admission of the inadequacy of the usual method of approach. By the use of many symbols and many equations the need for that pedagogic crutch is almost completely eliminated.

III. Other advantages follow from a thoroughgoing use of the multiple equation method:
1. The attention of the student is focused on the structure of the equation instead of on the component parts.
2. It is easier to check the equational translation against the words in the problem, for one has to look only at one sentence at a time.
3. It is simpler and easier to translate a problem consisting of two or more statements into the same number of equations rather than into one equation.
4. There is a great saving of time in not having to drill in the formation of algebraic phrases, since the equation (i.e. the sentence) is the unit of algebraic thought.
5. Less time is used up in the teaching of the so called type problems, for it is not necessary to teach the making of boxes, writing inscriptions above the boxes, etc. . . .
6. The use of the multiple equation method necessitates an earlier acquaintance with the process of substitution, which is more fundamental and of more general application than the customary solution by the multiplication-addition or the multiplication-subtraction method.
7. By introducing two and three unknowns in the solutions of the early problems one saves the two or three weeks that are usually allotted later in the course to the separate topic of simultaneous equations.
8. Perhaps the strongest argument for the proposed method is the superior power of analysis derived from the constant use of many symbols and many equations. The student thus forms a habit of reasoning and analysing in terms of symbols and relations which is the very core of mathematical thinking and which may carry over into other realms of his intellectual life.

2. Posey, L. R. A method of determining the sign and value of i^n where I equals $\sqrt{-1}$ and n is any rational positive integer equal to or greater than two. School Science and Mathematics. 34:812–15. November 1934.

The following are the rules given by the writer with proofs and examples for each:

a. $i^n = i$ if n be even and $n \div 2$ be even.
b. $i^n = i$, if n be odd and $(n-1) \div 2$ be even.
c. $i^n = -1$, if n be even and $n \div 2$ be odd.
d. $i^n = -i$, if n be odd and $(n-1) \div 2$ be odd.

Geometry

1. Carey, R. M. *Geometry in Secondary Schools*. The Mathematical Gazette. 18:217–222. October 1934.

The writer is keenly dissatisfied with parts of the present syllabus in geometry in secondary schools (of England), and suggests exactly what parts should be omitted and what retained.

"I noticed that congruent triangles needed more drill than any other part of geometry, and that it was more easily forgotten by boys." ". . . all profitable results (of congruent triangles) can be established by symmetry, a much shorter and more convincing method to men and boys." "The study of equivalent triangles fulfills no useful purpose; Pythagoras' Theorem can be established by other means."

Remarks of such a nature abound in the paper. Unfortunately, the rejections and recommendations seem to be based on personal taste only, and are not fortified by a mature and coherent theory of mathematical education.

2. Nyberg, Joseph A. *Remedial classes in geometry*. School Science and Mathematics. 34:853–61. November 1934.

An interesting report of the experiences of a teacher with a class in "remedial geometry," organized in the Hyde Park High School in Chicago for the slow, backward and dull pupils in that subject. Many of the observations that Mr. Nyberg makes are keen, and his comments are acute. But one hesitates to accept his conclusions. He believes "that more pupils fail from lack of trying than from lack of ability. The psychologist is always recommending easier courses for dull pupils. This assumes that the pupils fail from lack of ability. Too often it is merely lack of interest. What the pupil needs is not an easier course in geometry (which he would neglect), but more courses in woodwork, foundry and machine shops. School administrators know this and also know it is less expensive to organize a class in geometry than to equip a shop. In such a situation the teacher is helpless and can merely try to create an interest in geometry. But we ought to admit frankly that when giving courses we are doing something from necessity that should not be done at all."

Mr. Nyberg touches here on many moot problems in educational psychology and philosophy. Is it true that one who does poorly in geometry will necessarily do better work in carpentry or machine work? Or is that belief a hang over from a hoary psychology that contrasted brain and brawn to the disparagement of the latter? Even if we do admit that failure in geometry is "often due to lack of interest," will Mr. Nyberg deny that lack of interest is correlated very highly, in children at least, with lack of ability? And if we believe that geometry has something to offer of permanent value to every type of mind, why should not we offer an easy course in it for those who can not or who will not grasp a difficult one?

Miscellaneous

1. *La préparation théorique et practique des professeurs de mathématiques de l'enseignment secondaire dans les divers pays.* (*The theoretical and practical preparation of the teachers of mathematics in the secondary schools of various countries*). L'Enseignment

Mathématique. 32:5–22, 169–254, 360–400.

A detailed report of the committees appointed by the International Commission for the Teaching of Mathematics. The following countries coöperated in the above undertaking: the United States, Great Britain, France, Germany, Austria, Belgium, Hungary, Denmark, Italy, Norway, Poland, Switzerland, Chekoslovakia, and Jugoslavia.

It is intended to report from time to time in the subsequent issues of this department on the unusual features of the systems of mathematical education in foreign countries.

2. *Memorandum from the Mathematical Association (England) on the organization and the interrelation of mathematics in the secondary schools.* The Mathematical Gazette. 18:250–254. October 1934.

"Under the present system most schools enter as many pupils as possible for the School Certificate examinations. This inevitably leads to a standardized syllabus, and little regard can be paid to the *future* needs of the individual. It is felt that the present system is completely satisfactory only for that class of pupils who will proceed to a further education of university standard which includes mathematics as a main subject. For other classes the present position is not satisfactory:

a. Pupils who will proceed to a further education of a university type which does *not* include mathematics as a main subject,
b. Pupils who will proceed to a further education of a *technical* character (i.e. engineering) which includes mathematics as a main subject,
c. Pupils who will follow some fairly extended commercial course such as that required in banking or insurance,
d. Pupils who will proceed to a further education of a technical or commercial type in which mathematics is not a main subject,
e. Pupils whose education is not continued beyond the age of sixteen.

Each of these groups will undoubtedly contain some pupils able to reap full benefit from the present system, but it is felt nevertheless that some modification of the existing curriculum is desirable."

Specific recommendations are made for changes in the curriculum to suit the future needs of the above types. For pupils in (a), it is advised that a course should be given in the history of mathematics and in the practical applications of mathematics over as wide a field as possible. The whole object would be to indicate the place of mathematics in modern life and thought.

For pupils in (b) and (c), it is urged that they take up the practical part of the subject at an earlier stage, even at a sacrifice of some of the more academic parts of the syllabus.

The pupils in (d) and (e), though they seem unable to deal with that part of the work requiring much logical argument, are as a rule capable of good work in the more straightforward parts of arithmetic, the bare elements of algebra, mensuration, practical geometry and in numerical trigonometry. It is recommended that a full course of the above scope should be allowed them.

Recommendations are also made in reference to the coöperation between teachers of different subjects and teachers of mathematics. The training of teachers is also discussed.

3. Reeve, W. D. *Correlation and secondary school mathematics.* National Mathematics Magazine. 9:10–12. October 1934.

After pointing out that the subject of mathematics and those who teach it

are coming in for an appreciable share of criticism, Dr. Reeve makes two recommendations:

a. The teacher of mathematics in the United States should be held to a higher standard of scholarship.
b. There should be more correlation between mathematics and other fields of human endeavor, and between the various branches of mathematics itself. "Mathematics has contributions to make to other subjects like biology, astronomy, physics, chemistry, and engineering. The problem for the teacher is to equip himself as well as he can in all of these fields so that he can bring out the proper correlation when the time comes." "We need to show how the study of mathematics will help one to be a better student of science, music or the arts. We need also to teach mathematics as a method of thinking so that the pupils who sit at our feet will go out into life able to think better because they have learned to think in studying their mathematics."

4. Rothe, Rudolph. *Mathematics and military science.* (*Mathematik und Wehrwissenschaft*). Unterrichtsblätter für Mathematik und Naturwissenschaften. 40:195–204. 1934 no. 6.

A glowing account of the application of mathematics to the art of making war. The author shows, with painstaking detail, the mathematical problems that are involved,

a. in surveying and map-making,
b. in the location of enemy forts and snipers, and
c. in the determination of the range of guns and the optimum angle at which they should be adjusted

This article should serve as a crushing answer to the common complaint that mathematics has no application to problems of life!

5. Shaw, James B. *Mathematics—the subtle fine art.* The Scientific Monthly 39:420–33. November 1934.

Another answer to the perennial questions: "What is mathematics?" and "Why should mathematics be of interest to any but a few?"

"Mathematics is on the artistic side a creation of new rhythms, orders, designs and harmonies, and on the knowledge side is a systematic study of various rhythms, orders, designs and harmonies." Mathematics should be studied "not, of course, for the mere scholarship involved, not for the keen intelligence it will promote but for the high order of imagination it will demand, for the incisive artistic insight it will generate."

The above analysis of the value of mathematics should, however, remain *entre nous*. It would be best not to offer it as a reason for the teaching of mathematics, to a chamber of commerce crying for a decrease in the tax rate!

6. Walker, Helen M. *Abraham De Moivre.* Scripta Mathematica. 2:316–333. August 1934.

A biographical study of a celebrated mathematician which is not only a record of the greatness of his genius but a re-creation of a fascinating personality as well.

Teachers of trigonometry and advanced algebra would do well by assigning this article as collateral reading in connection with the study of the famous theorem that bears his name.

Registration at the National Council Meeting

As the 1935 Annual Meeting of the National Council of Teachers of Mathematics at Atlantic City, N.J., approaches, the secretary is very anxious that the members, guests and visitors register properly. Four different colored registration cards are used to classify our registration.

1. *Visitors* and *guests*, that is *non-members* register on green cards.
2. *Members* who are attending their *first* annual meeting register on white cards.
3. *Members* who have attended more than one meeting but less than half of the annual meetings register on a red card.
4. *Members* who have attended *half or more* of the annual meetings register on a blue card, indicating thereby their "true blue" loyalty to the organization. A member does not continue in the blue class unless he maintains his record of attending at least half of the annual meetings. If he does not measure up he slips back into the red class.

The following list indicates when and where our annual meetings have been held:

1920—Cleveland	1925—Cincinnati	1930—Atlantic City
1921—Atlantic City	1926—Washington	1931—Detroit
1922—Chicago	1927—Dallas	1932—Washington
1923—Cleveland	1928—Boston	1933—Minneapolis
1924—Chicago	1929—Cleveland	1934—Cleveland

Before you go to Atlantic City will you decide which of the annual meetings you have attended and thus facilitate registration by asking for the proper colored registration card. Persons attending any sessions of *The Council* should register so that the secretary will have his records complete.

EDWIN W. SCHREIBER, *Secretary*

Be Sure to Vote the Ballot on Page 1 of This Issue